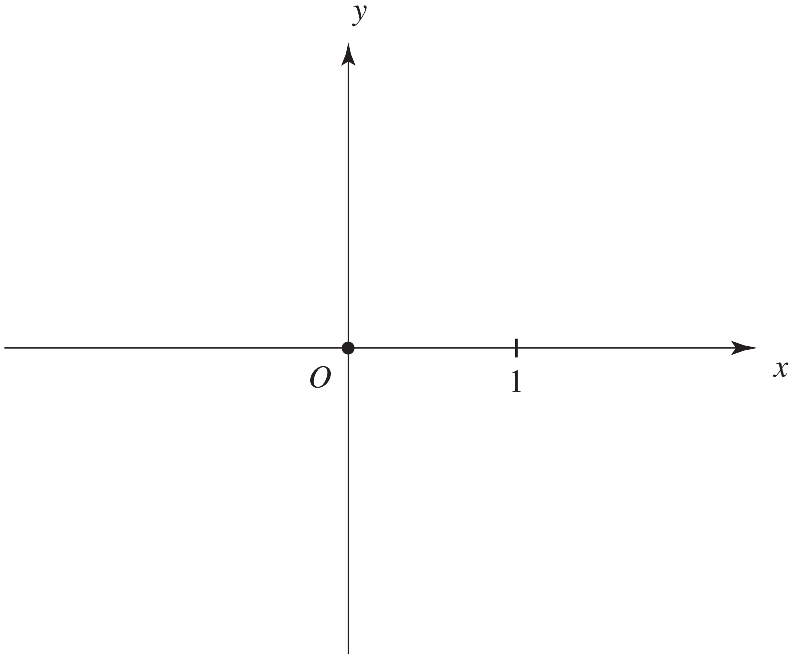
Exam **Name**

Please Note: Calculators may be used in elementary and trig modes, but not in Calculus mode.

1. Find a function $y = f(x)$ that satisfies the differential equation $y\frac{dy}{dx} = \frac{\ln x}{e^{y^2}}$ and the initial condition $f(1) = 0$.

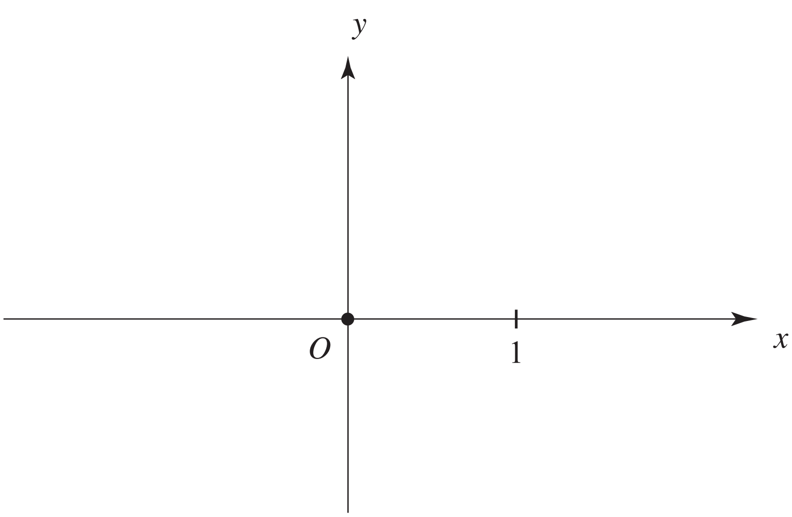
2. Find a function $y = f(x)$ that satisfies the differential equation $y' + \frac{1}{x}y = x^2$ and the initial condition $f(1) = 1$.

3. A complex plane given by a polar coordinate system and expanded to include the real and imaginary axes is shown below. Place the complex numbers $c_1 = (2, \frac{\pi}{6}), c_2 = (1, \frac{-\pi}{4})$ and $c_3 = (-\frac{1}{2}, \frac{3\pi}{4})$ in the plane. Determine the polar coordinates of c_2c_3 and put it into the plane. Express the numbers c_1 and c_2c_3 in the form $a+bi$ and compute $c_1+c_2c_3$. Put the result into the box below.



$\boxed{}$

4. Specify a complex number c in polar coordinates that satisfies $c^5 = -1$ (with $c \neq -1$) and place it into the complex plane below. Explain why your answer c satisfies $c^5 = -1$. Determine the Cartesian coordinates of this point in the plane and rewrite c in the form $c = a + bi$.



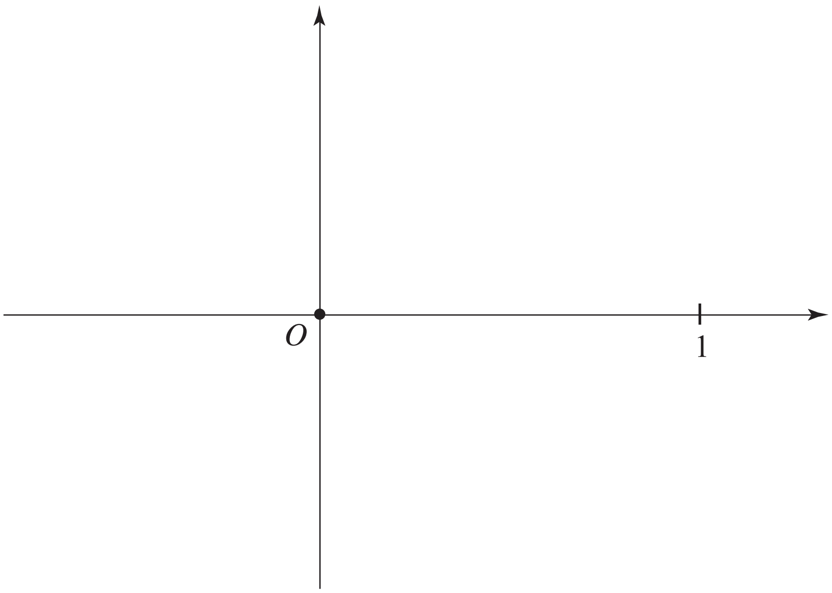
$c = ($, $)$ in polar $c = ($ $) + ($ $)\, i$ in Cartesian

5. Find a function $y = f(x)$ that satisfies the differential equation $y'' + 2y' + 5y = 0$ as well as the conditions $f(\frac{\pi}{4}) = 0$ and $f'(\frac{\pi}{2}) = e^{-\frac{\pi}{2}}$. Put the resulting $y = f(x)$ into the box below.

6. Consider the ellipse $\frac{x^2}{7^2} + \frac{y^2}{4^2} = 1$. Find a polar function of the form $r = f(\theta) = \frac{d}{1+\varepsilon\cos\theta}$ with the property that its graph has the same shape as that of the given ellipse. Put your answer into the box.

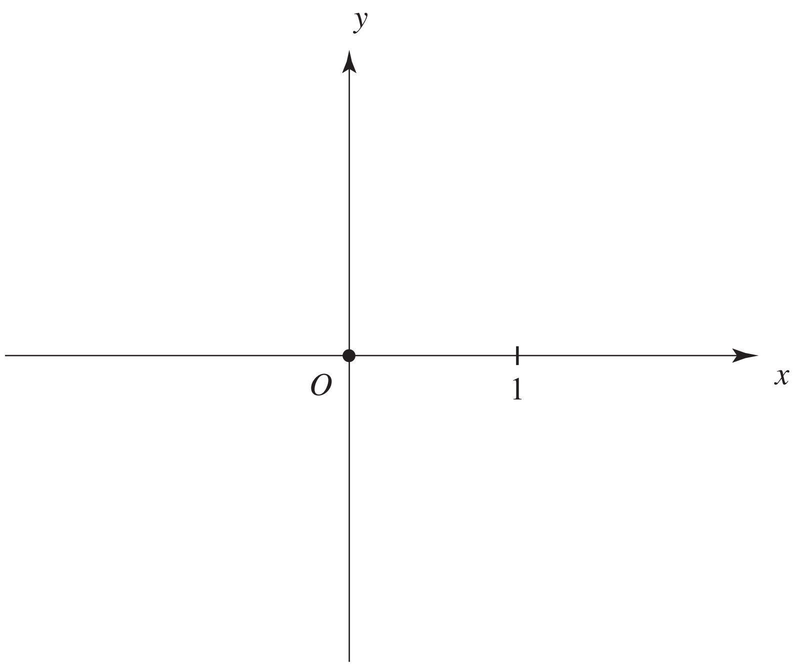
7. Consider the function $r = f(\theta) = \cos\theta$. The plane below has an xy-coordinate system superimposed over a polar coordinate system.

i. By going from polar to Cartesian coordinates verify that the graph of $r = f(\theta)$ is a circle. Sketch the circle into the plane.



ii. Pick a general point P on the circle. Draw the tangent line to the circle at P and draw in the corresponding angle α. Make use of a property of isosceles triangles to verify that $f'(\theta) = \tan(\alpha - \frac{\pi}{2}) \cdot f(\theta)$ holds for the function $r = f(\theta) = \cos\theta$.

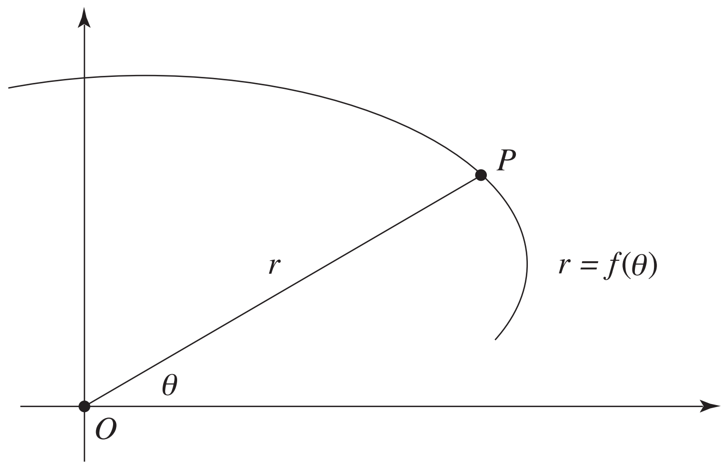
8. Sketch the graph of the polar function $r = f(\theta) = \frac{1}{\cos\theta}$ into the coordinate plane below.



Make use of the graph to compute the integral $\int_0^{\frac{\pi}{3}} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$. and put your answer into the box below.

Then use the graph compute the integral $\int_0^{\frac{\pi}{3}} \frac{1}{2} f(\theta)^2 \, d\theta$ and put your answer into the box.

9. The figure below shows a planet in motion around the Sun. The x and y coordinates are both differentiable functions of t and the derivatives $\frac{dx}{dt} = x'(t)$ and $\frac{dy}{dt} = y'(t)$ are differentiable as well. One of the early facts in the modern theory of gravitation is the equality $x\,\frac{dy}{dt} - y\,\frac{dx}{dt} = c$, where c is a constant.



i. Switch to polar coordinates (r, θ) and let $r = f(\theta)$ be a function that has the planet's trajectory as its graph. Use the fact that $\theta = \theta(t)$ and $r(t) = f(\theta(t))$ are both differentiable functions of t, to show that $r(t)^2\theta'(t) = c$.

ii. Consider P at two different times t_1 and t_2 in its orbit and suppose that $t_1 < t_2$. Explain the meaning of the integral $\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \sqrt{f'(\theta(t))^2 + f(\theta(t))^2}\,\theta'(t)dt$.

Formulas and expressions: $\displaystyle\int u\,dv = uv - \int v\,du$ $\quad Ay'' + By' + Cy = 0$ $\quad y = D_1 e^{r_1 x} + D_2 e^{r_2 x}$
$y = D_1 e^{2x} + D_2 x e^{2x}$ $\quad y = e^{ax}(D_1 \cos bx + D_2 \sin bx)$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$ $\quad x = r\cos\theta$ $\quad y = r\sin\theta$
$a = \frac{d}{1-\varepsilon^2}$ $\quad b = \frac{d}{\sqrt{1-\varepsilon^2}}$ $\quad a = \frac{d}{\varepsilon^2-1}$ $\quad b = \frac{d}{\sqrt{\varepsilon^2-1}}$ $\quad f'(\theta) = f(\theta)\cdot\tan(\alpha - \frac{\pi}{2})$ $\quad \displaystyle\int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$
$\frac{1}{2}r^2\theta$ $\quad \displaystyle\int_a^b \tfrac{1}{2}\,f(\theta)^2\,d\theta$